MA10209 – Week 2 Tutorial

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Top Tips (response to sheet 1)

- Make sure you answer the question -

  Does this information determine $|A \cup B \cup C|$?

  requires a yes/no as well as an explanation.
- Explain your answers in words.
  - Don't rely on diagrams.
- Be careful not to miss parts of the question accidentally.

Top Tips (response to sheet 1)

- Equals means equals.
  - Don't set functions equal to real numbers
    ~~$f(x) = x = \mathrm{Id}_{\mathbb{R}}$~~ is a nonsense statement!
- Consider the difference between infinitely many maps and one map taking infinitely many values.
  - For example,

$$f : \mathbb{Z} \to \mathbb{Z}, \qquad x \mapsto x^3$$

is a single map, but

$$g_n : \mathbb{Z} \to \mathbb{Z}, \qquad x \mapsto x^n \qquad \text{for } n \in \mathbb{N}$$

is infinitely many maps.

Top Tips (response to sheet 1)

- Know when to use a proof and when to use a counterexample.
  - A counter-example shows a statement doesn't hold,
    - "All prime numbers are odd."
  - but sometimes proving the converse is easier.
    - "There are infinitely many even prime numbers."
  - A proof is required to show a statement holds.
    - "There is exactly one even prime number."

- Try not to include irrelevant information in your answer.

Functions/Maps: definition

- What do we need for a function/map?

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  - Domain
  - Codomain
  - Rule which assigns to each element of the domain a single element of the codomain

Functions/Maps: properties

Let $f : A \to B$ be a map.

- What do the following special properties mean?

Injective	Surjective

Bijective

Functions/Maps: properties

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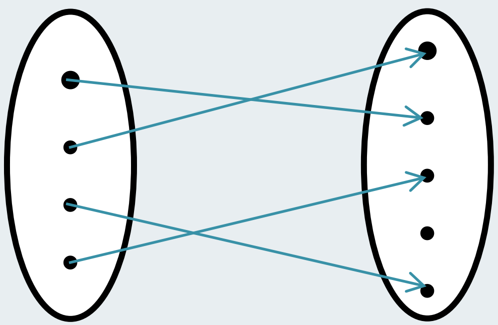
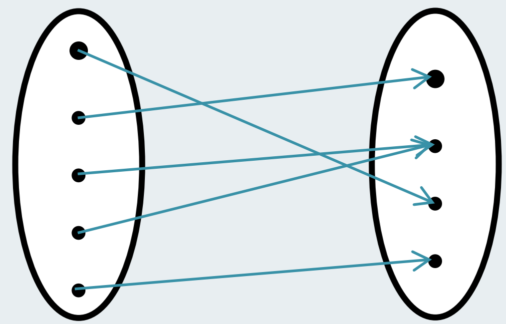
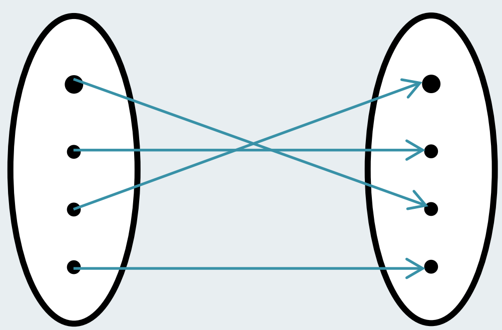
Injective



Surjective



Bijective



Functions/Maps: properties

Let $f : A \to B$ be a map.

- What do the following special properties mean?

Injective	Surjective
$f(x) = f(y) \Rightarrow x = y$	For $z \in B \quad \exists x \in A$ s.t. $f(x) = z$

Bijective

f is both injective and surjective

Functions/Maps: examples

- Are the following (a) injective, (b) surjective, (c) bijective?

$$f : \mathbb{R} \to \mathbb{R} \qquad x \mapsto \begin{cases} 1/x & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

$$g : \mathbb{R} \to \mathbb{R} \qquad x \mapsto \sin(x)$$

Functions/Maps: examples

- Is the following (a) injective, (b) surjective, (c) bijective?

$$f : \mathbb{R} \to \mathbb{R} \qquad x \mapsto \begin{cases} 1/x & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Injective?

Assume $f(x) = f(y)$.
If $x = 0$, then $f(y) = f(x) = 0 \Rightarrow y = 0$.
Otherwise $x \neq 0$. In this case,

$$\begin{aligned} f(x) &= f(y) \\ \frac{1}{x} &= \frac{1}{y} \\ x &= y \qquad \text{since } x, y \neq 0 \end{aligned}$$

So f is injective.

Functions/Maps: examples

- Is the following (a) injective, (b) surjective, (c) bijective?

$$f : \mathbb{R} \to \mathbb{R} \qquad x \mapsto \begin{cases} 1/x & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Surjective?

Consider $z \in \mathbb{R}$.
If $z = 0$, then we have $x = 0 \in \mathbb{R}$
with $f(x) = f(0) = 0 = z$.

Otherwise $z \neq 0$.
Set $x = \frac{1}{z}$ and notice that $x = \frac{1}{z} \in \mathbb{R}$.
Then $f(x) = f(\frac{1}{z}) = z$.
So f is surjective.

Functions/Maps: examples

- Is the following (a) injective, (b) surjective, (c) bijective?

$$f : \mathbb{R} \to \mathbb{R} \qquad x \mapsto \begin{cases} 1/x & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Bijective?

Function f is both injective and surjective, so it is bijective.

Functions/Maps: examples

- Is the following (a) injective, (b) surjective, (c) bijective?

$$g : \mathbb{R} \to \mathbb{R} \qquad x \mapsto \sin(x)$$

Injective?

Notice that $g(0) = g(\pi)$, but $0 \neq \pi$.

Functions/Maps: examples

- Is the following (a) injective, (b) surjective, (c) bijective?

$$g : \mathbb{R} \to \mathbb{R} \qquad x \mapsto \sin(x)$$

Surjective?

$$\text{Consider } 2 \in \mathbb{R}.$$

$$\text{Since } \forall y \in \mathbb{R}, \sin(y) \leq 1,$$

$$\nexists x \in \mathbb{R} \text{ with } \sin(x) = 2.$$

Creating bijections

- Give a bijection between the following two sets:

$$A = \{r | r \in \mathbb{N}, r > 5\}$$

$$B = \{2r | r \in \mathbb{N}\}$$

$$A = \mathbb{N}$$

$$B = \{r | r \in \mathbb{N}, r \text{ is prime}\}$$

Creating bijections

- Give a bijection between the following two sets:

$$A = \{r | r \in \mathbb{N}, r > 5\}$$
$$B = \{2r | r \in \mathbb{N}\}$$

  - One possible answer:

$$f : A \rightarrow B \qquad x \mapsto 2(x - 5)$$

  - Why is this a bijection?

Creating bijections

- Give a bijection between the following two sets:

$$A = \{r | r \in \mathbb{N}, r > 5\}$$
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  - One possible answer:

$$f : A \to B \qquad x \mapsto 2(x - 5)$$

  - Why is this a bijection?

$$f = h \circ g \text{ where}$$
$$g : A \to \mathbb{N} \qquad x \mapsto x - 5$$
$$h : \mathbb{N} \to B \qquad x \mapsto 2x$$

Creating bijections

- Give a bijection between the following two sets:

$$A = \mathbb{N}$$

$$B = \{r | r \in \mathbb{N}, r \text{ is prime}\}$$

  - One possible answer:

$$f : A \to B \qquad n \mapsto \text{n}^{\text{th}} \text{ prime number}$$

  - Why is this a bijection?

Counting maps

$$\text{Let } A = \{1,2\} \text{ and } B = \{1,2,3\}.$$

- a) How many maps are there from A to B?
- b) How many injective maps are there from A to B?
- c) How many surjective maps are there from A to B?

- d) How many maps are there from B to A?
- e) How many injective maps are there from B to A?
- f) How many surjective maps are there from B to A?

Counting maps

$$\text{Let } A = \{1, 2\} \text{ and } B = \{1, 2, 3\}.$$

- a) How many maps are there from A to B? 9
- b) How many injective maps are there from A to B? 6
- c) How many surjective maps are there from A to B? 0

- d) How many maps are there from B to A? 8
- e) How many injective maps are there from B to A? 0
- f) How many surjective maps are there from B to A? 6

Exercise Sheet 2 - overview

- Q1 – lots of problems. If it's not a map, clearly state what's wrong with it.
  - (b) consider $z = -1$.
- Q2 – if the property holds, prove it does. If it doesn't, find a counter-example
  - (c) & (d) $|a + bi| = \sqrt{a^2 + b^2} \geq 0.$
  - (e) explain!
  - (f) consider graph of $\tan(x)$.
- Q3 – considering smaller examples may help find the solution. Remember to justify answers.

Exercise Sheet 2 - overview

- Q4 – select the right maps, and everything is reasonably obvious (hopefully)
- Q5 – remember how to prove injectivity and surjectivity. Use counterexamples if they're not true.

Note: it is a good idea to try to find counterexamples on small sets. This will help you think about exactly where it breaks.

Exercise Sheet 2 - overview

- Q6
  - (a) & (b) use question 5
  - (c) note that $f \circ g \circ f = (f \circ g) \circ f = f \circ (g \circ f)$.
- Q7
  - (a) fix $h(0) = y$ for some $y \in \mathbb{R}$. Show that this is enough to identify the map by using induction (remember you're working on all of $\mathbb{Z}$).

    This give one direction:

    $$f \circ h = h \circ f \quad \Rightarrow \quad h(x) = \ldots$$

    You also need to show the other.

Exercise Sheet 2 - overview

- Q8 – quite tricky. It's designed to get you thinking about the necessary properties of a bijection, and about creative ways to create them.
  - (c) Find a bijection between $\{r | r \in \mathbb{R}, 0 < r < 1\}$ and $\{r | r \in \mathbb{R}, -\frac{\pi}{2} < r < \frac{\pi}{2}\}$. From there, previous questions may be useful!
  - (d) Try to find a way to represent each finite subset of $\mathbb{N}$ and that will give you a unique number for each subset.
- Q9 – If we have n+1 terms and n possible values, then there must be a repeat somewhere…