MA10209 – Week 3 Tutorial

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How to go about a proof

- Step 1: Identify what you need to show
  - Sometimes this is obvious from the question, other times you'll need to look up a definition.
  - If appropriate, break your proof down into parts.
    E.g. to show that something is bijective, it might help to do it in two stages: (1) injective, then (2) surjective.

How to go about a proof

Let $f : A \to B$ and $g : B \to C$.
Suppose that $g \circ f$ is injective.
Show that f is injective.

To show that f is injective we need:
For $x_1, x_2 \in A$,
$f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.

How to go about a proof

- Step 2: Work out where to start
  - If you have to prove that $\text{“Statement 1”} \Rightarrow \text{“Statement 2”}$ then you start with statement 1, and work your way to statement 2.
  - If you have to prove that $\text{“Statement 1”} \Leftrightarrow \text{“Statement 2”}$ then you need to complete two proofs, one starting at each side.

How to go about a proof

Let $f : A \to B$ and $g : B \to C$.
Suppose that $g \circ f$ is injective.
Show that f is injective.

To show that f is injective we need:
For $x_1, x_2 \in A$,
$f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.

Assume we have $x_1, x_2 \in A$
such that $f(x_1) = f(x_2)$.

How to go about a proof

- Step 3: Identify relevant definitions and statements
  - Have your notes open in front of you!
  - Write what the definitions are using the symbols you are given in the question.

How to go about a proof

Let $f : A \to B$ and $g : B \to C$.

Suppose that $g \circ f$ is injective.

Show that f is injective.

Assume we have $x_1, x_2 \in A$

such that $f(x_1) = f(x_2)$.

Now since $g \circ f$ is injective, we know that

$$g \circ f(x_1) = g \circ f(x_2) \Rightarrow x_1 = x_2$$

How to go about a proof

- Step 4: Look for things that look like either where you're coming from, or where you're going to.
  - Can you see a result from lectures that would give you what you need provided you can satisfy certain conditions?
  - Does one of the conditions in the question help you progress in your answer?

  - This is the difficult bit – you might need to try a few ideas before finding the right answer.

How to go about a proof

$$\text{Let } f : A \to B \text{ and } g : B \to C.$$

$$\text{Suppose that } g \circ f \text{ is injective.}$$

Show that f is injective.

$$\text{Assume we have } x_1, x_2 \in A$$

$$\text{such that } f(x_1) = f(x_2).$$

What conditions are necessary for this step?

$$\text{Now since } g \circ f \text{ is injective, we know that}$$

$$g \circ f(x_1) = g \circ f(x_2) \Rightarrow \boxed{x_1 = x_2}$$

Here's what we want to show

How to go about a proof

- Step 5: Write the proof carefully, showing the conditions hold where necessary.
  - If you use a result from lectures or another question on the sheet,
    - (a) make it clear where the result comes from, and
    - (b) state why you can use it, being careful with symbols if the ones you're using are different.

How to go about a proof

Let $f : A \to B$ and $g : B \to C$.
Suppose that $g \circ f$ is injective.
Show that f is injective.

Assume we have $x_1, x_2 \in A$
such that $f(x_1) = f(x_2)$.
Then $g(f(x_1)) = g(f(x_2))$ and so
$g \circ f(x_1) = g \circ f(x_2)$.
Since $g \circ f$ is injective, this gives
$x_1 = x_2$, so f is injective.

Top Tips (response to sheet 2)

- Make sure what you write makes sense.

$$x = y \quad x \sim y \qquad x = x \quad x \sim x$$

Once you've finished writing, try to read it out loud.

$$\text{For } x, y \in \mathbb{R}, \text{ say } x \sim y \text{ iff } x = y.$$
$$\text{Since } \forall x \in \mathbb{R},\ x = x, \text{ we have } x \sim x.$$
$$\text{So } \sim \text{ is reflexive.}$$

Top Tips (response to sheet 2)

- Be careful to say what you mean
  - The following doesn't make sense:

$$-1 \neq \mathbb{N}$$

since the left-hand side is a number and the right-hand side is a set, and we haven't defined when a number is equal to a set.

Top Tips (response to sheet 2)

- Be specific with counterexamples, and general with proofs
  - A proof needs to cover every available case, but
  - you only need the statement to break at one value for it to be false, so make it absolutely clear by choosing a specific case.

Definitions to look at

- Set X countable
- Power set $\mathcal{P}(X)$
- Partition
- Reflexive
- Symmetric
- Transitive
- Equivalence relation
- Equivalence classes
- Transversal

Reflexive, Symmetric, Transitive

Say $x \star y$ iff…	reflexive	symmetric	transitive
	✗	✗	✗
	✓	✗	✗
	✗	✓	✗
	✗	✗	✓
	✓	✓	✗
	✗	✓	✓
	✓	✗	✓
	✓	✓	✓

Reflexive, Symmetric, Transitive

- Some ideas of relations

Let $x, y \in \mathbb{Z}$, and say $x \sim y$ iff...

$$x < y$$

$$x = y$$

$$x - y \leq 1$$

$$x \neq y$$

$$x \leq y$$

$$x - y = 1$$

$$x, y \text{ both even}$$

$$|x - y| \leq 2$$

Reflexive, Symmetric, Transitive

Say $x \star y$ iff…	reflexive	symmetric	transitive
$x - y = 1$	✗	✗	✗
$x - y \leq 1$	✓	✗	✗
$x \neq y$	✗	✓	✗
$x < y$	✗	✗	✓
$\lvert x - y \rvert < 2$	✓	✓	✗
x, y both even	✗	✓	✓
$x \leq y$	✓	✗	✓
$x = y$	✓	✓	✓

Equivalence relations

$$\text{Let } x, y \in \mathbb{Z} \text{ and say } x \sim y \text{ iff } x - y \text{ is even.}$$

Show '$\sim$' is an equivalence relation.

- **Reflexive:**

  Since for all $x \in \mathbb{Z}$, $x - x = 0$, we have $x \sim x$

- **Symmetric:** Let $x \sim y$, then $x - y$ is even.

  Since $x - y$ is even, then so is $y - x = -(x - y)$, so $y \sim x$.

- **Transitive:**

  Let $x \sim y$ and $y \sim z$, that is $x - y$ and $y - z$ are even.

  Then $x - z = x - y + y - z = (x - y) + (y - z)$ is even, so $x \sim z$.

Partitions

$$\text{Consider the set } \{1,2,3,4,5,6\}.$$

- How many ways are there to partition this into three subsets?
  - Shape (4,1,1)
    $$\binom{6}{4} = 15$$
  - Shape (3,2,1)
    $$\binom{6}{3}\binom{3}{2} = 60$$
  - Shape (2,2,2)
    $$\frac{\binom{6}{2}\binom{4}{2}\binom{2}{2}}{3!} = 15$$

Transversals

$$\text{Consider the set } \{1, 2, 3, 4, 5, 6\}$$
$$\text{partitioned into } \{1, 3\}, \{2, 4, 6\}, \{5\}.$$

- What are all the transversals of this partition?

  (There should be 6)

Exercise Sheet 3 - overview

- Q1 – play with lots of definitions
  - (d) similar to Sheet 2 Q8(b)
- Q2 – remember you're only looking for sizes – don't try to list all the sets!
- Q3 – make a list of the possible shapes of the partitions, then figure out how many there are of each shape
- Q4 – say whether it's reflexive, symmetric and/or transitive, and give explanations.

Exercise Sheet 3 - overview

- Q6
  - (b) $|e^{i\theta}| = 1 \text{ for all } \theta \in [0, 2\pi)$
- Q8
  - It is a lot easier to show this is reflexive and symmetric than it is to show it's transitive
  - Be careful how you label sets, try not to get confused between them
- Q9
  - Part (a) is just more working through the definitions